

MINIMUM-ENERGY PRINCIPLE FOR A FREE-BOUNDARY, FORCE-FREE PLASMA

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Abstract—A variational principle for a cold plasma, formulated by TAYLOR (1974), minimizes the magnetic energy of a cylindrical plasma confined by an infinitely conducting metal wall, subject to the constraints of constant longitudinal flux and magnetic helicity. The paper extends this minimum energy principle to the case of a vacuum layer separating the plasma from the wall. The ensuing shift of the F - β -curve is in qualitative agreement with the experiments.

1. INTRODUCTION

A VARIATIONAL principle by TAYLOR (1974) minimizes the magnetic energy of a cylindrical plasma column enclosed by an infinitely conducting metal wall, subject to the constraints of constant longitudinal flux and magnetic helicity $K = \int \mathbf{A} \cdot \mathbf{B} \, d\tau$ (Lagrange multiplier $-\lambda/2$). The minimizing magnetic field is a solution of the Euler equation

$$\nabla \times \mathbf{B} = \lambda \mathbf{B}, \quad (1)$$

and its Bessel solution correctly demonstrates the reversal of the longitudinal field component, observed in reversed field pinches.

Minor shortcomings of this theory are its limitation to low- β (cold) plasmas and its inability to describe plasma equilibria with a current density that falls off rapidly towards the metal wall. To the latter problem, a number of solutions have already been proposed (ORTOLANI *et al.*, 1984; HASEGAWA *et al.*, 1984; SCHOENBERG *et al.*, 1984). In some of these solutions, the role of a vacuum layer between the plasma and the metal wall is acknowledged without, however, incorporating the vacuum layer into the variational principle. In the underlying paper, the total magnetic energy (plasma + vacuum) is minimized subject to the plasma magnetic helicity and other constraints, necessary to let the occurring boundary terms in the principle vanish.

2. THE VARIATIONAL PRINCIPLE

(a) Problem formulation

We consider a toroidal plasma column surrounded by a vacuum layer and an infinitely conducting metal wall with circular cross-section. The plasma boundary is allowed to move during the relaxation. In the variational principle, both the vector potential and the position of the plasma edge are varied and acquire stationary values at minimum total energy of the configuration. The constraints needed in this model are the conservation of the toroidal fluxes in the plasma and in the vacuum region,

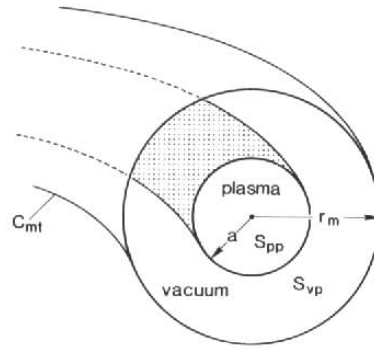


FIG. 1.—Geometry of the model.

the magnetic helicity of the plasma and the poloidal vacuum flux. The latter constraint arises from the assumption of infinite conductivity of the plasma–vacuum interface. The constraint can be omitted if a surface current at the plasma boundary is excluded.

The mathematical formulation becomes as follows. We minimize the total magnetic energy

$$W_B = \int_{V_p} \frac{B_p^2}{2\mu_0} dV + \int_{V_v} \frac{B_v^2}{2\mu_0} dV, \quad (2)$$

with respect to variations in V_p , $\mathbf{A}_p$ and $\mathbf{A}_v$. V_p and V_v are the volumes of the plasma and the vacuum region respectively (see Fig. 1), and $\mathbf{A}_p$ and $\mathbf{A}_v$ are the magnetic vector potentials of the plasma and vacuum regions respectively. The conserved quantities are:

$$\text{magnetic helicity of the plasma} \quad \mu_0 K = \int_{V_p} \mathbf{A}_p \cdot \mathbf{B}_p dV, \quad (3)$$

$$\text{toroidal plasma flux} \quad \psi_{pt} = \int_{S_{pp}} \mathbf{B}_p \cdot d\mathbf{S}, \quad (4)$$

$$\text{toroidal vacuum flux} \quad \psi_{vt} = \int_{S_{vp}} \mathbf{B}_v \cdot d\mathbf{S}, \quad (5)$$

$$\text{poloidal vacuum flux} \quad \psi_{vp} = \int_{S_{vt}} \mathbf{B}_v \cdot d\mathbf{S}. \quad (6)$$

The toroidal geometry calls for an additional constraint, the flux through the hole of the torus,

$$\psi_h = \int_{S_h} \mathbf{B}_h \cdot d\mathbf{S} = \int_{C_{mt}} \mathbf{A}_v \cdot d\mathbf{C}. \quad (7)$$

Furthermore, there are a number of boundary conditions. First, we have the continuity of the vector potential and of the magnetic pressure across the plasma

boundary $\mathbf{r}_b$:

$$\mathbf{A}_p(\mathbf{r}_b) = \mathbf{A}_v(\mathbf{r}_b), \quad (8)$$

$$\frac{1}{2\mu_0} B_p^2(\mathbf{r}_b) = \frac{1}{2\mu_0} B_v^2(\mathbf{r}_b). \quad (9)$$

Secondly, since the plasma boundary and the metal surface are flux surfaces, we have

$$\mathbf{n}_p(\mathbf{r}_b) \cdot \mathbf{B}_p(\mathbf{r}_b) = 0, \quad (10)$$

$$\mathbf{n}_m(\mathbf{r}_m) \cdot \mathbf{B}_v(\mathbf{r}_m) = 0. \quad (11)$$

(b) *The first variation of W_B*

The first variation δW_B can be written as $W_{B1} - W_{B0}$, where W_{B0} is the magnetic energy in the sought minimum state and W_{B1} is the energy in the perturbed state, a small change of plasma volume included. Giving all minimum quantities the index 0 and the perturbed quantities the index 1, we have (see Fig. 2):

$$\delta W_B = \int_{V_{p1}} \frac{B_{p1}^2}{2\mu_0} dV - \int_{V_{p0}} \frac{B_{p0}^2}{2\mu_0} dV + \int_{V_{v1}} \frac{B_{v1}^2}{2\mu_0} dV - \int_{V_{v0}} \frac{B_{v0}^2}{2\mu_0} dV. \quad (12)$$

Defining likewise in volume V_{p0}

$$\delta \frac{B_p^2}{2\mu_0} = \frac{B_{p1}^2}{2\mu_0} - \frac{B_{p0}^2}{2\mu_0}, \quad (13a)$$

and in volume V_{v0}

$$\delta \frac{B_v^2}{2\mu_0} = \frac{B_{v1}^2}{2\mu_0} - \frac{B_{v0}^2}{2\mu_0}, \quad (13b)$$

we can continue the functions and their variations analytically into the increment volume ΔV_p .

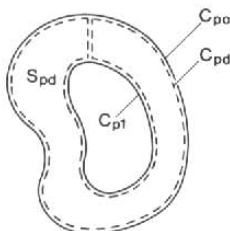


FIG. 2.—Perturbed (C_{p1}) and unperturbed (C_{p0}) poloidal contours of the plasma boundary. C_{pd} is the contour used in equations (17) and (18).

For a small displacement $\delta \mathbf{a}$ of the plasma surface S_{p0} , the integral over the volume V_{p1} can be written as:

$$\int_{V_{p1}} \frac{B_{p1}^2}{2\mu_0} dV = \int_{V_{p0}} \frac{B_{p1}^2}{2\mu_0} dV + \int_{S_{p0}} \frac{B_{p1}^2}{2\mu_0} \delta \mathbf{a} \cdot d\mathbf{S}, \quad (14)$$

an analogous relation holding for the vacuum integral. With the equations (13) and (14), equation (12) becomes

$$\delta W_B = \int_{V_{p0}} \delta \frac{B_p^2}{2\mu_0} dV + \int_{V_{v0}} \delta \frac{B_v^2}{2\mu_0} dV + \int_{S_{p0}} \left(\frac{B_{p0}^2}{2\mu_0} - \frac{B_{v0}^2}{2\mu_0} \right) \delta \mathbf{a} \cdot d\mathbf{S}, \quad (15)$$

where $d\mathbf{S}$ is in the direction of the outward normal to the plasma surface or to the metal wall.

(c) *The total variation $\delta W_B - \lambda/2 \delta K$*

We take the variation of equation (3) and transform all variations $\delta \mathbf{B}$ appearing in the volume integrals to $\delta \mathbf{A}$ by partial integrations. The final result is

$$\begin{aligned} \delta W_B - \frac{\lambda}{2} K &= \int_{V_p} \delta \mathbf{A}_p \cdot (\nabla \times \mathbf{B}_p - \lambda \mathbf{B}_p) dV + \int_{V_v} \delta \mathbf{A}_v \cdot \nabla \times \mathbf{B}_v dV \\ &+ \int_{S_{p0}} \left\{ \frac{1}{2} (B_p^2 - \lambda \mathbf{A}_p \cdot \mathbf{B}_p - B_v^2) \delta \mathbf{a} + \delta \mathbf{A}_p \times \left(\mathbf{B}_p - \frac{1}{2} \lambda \mathbf{A}_p \right) - \delta \mathbf{A}_v \times \mathbf{B}_v \right\} \cdot d\mathbf{S}_p \\ &+ \int_{S_m} (\delta \mathbf{A}_v \times \mathbf{B}_v) \cdot d\mathbf{S}_m = 0, \quad (16) \end{aligned}$$

where S_m is the metal surface, and $d\mathbf{S}_m$ is in the direction of the inward normal to the metal surface.

There exists a relation between the variation $\delta \mathbf{A}$ and the variation $\delta \mathbf{a}$ of the position of the plasma boundary. This relation is derived in the following way.

In Fig. 2 the integral along the perturbed contour C_{p1} of the poloidal cross-section can be written as the integral along the equilibrium contour C_{p0} plus a correction term due to the distortion of the contour:

$$\int_{C_{p1}} \mathbf{A}_{p1} \cdot d\mathbf{C} = \int_{C_{p0}} \mathbf{A}_{p1} \cdot d\mathbf{C} + \int_{C_{pd}} \mathbf{A}_{p1} \cdot d\mathbf{C}, \quad (17)$$

where C_{pd} denotes the dashed contour. According to Stokes' theorem

$$\int_{C_{pd}} \mathbf{A}_{p1} \cdot d\mathbf{C} = \int_{S_{pd}} \mathbf{B}_{p1} \cdot d\mathbf{S}, \quad (18)$$

with the infinitesimal surface element $d\mathbf{S} = \delta\mathbf{a} \times d\mathbf{C}$. Since $\delta\mathbf{a}$ is of first order, $\mathbf{B}_{p1}$ in equation (18) may be replaced by the equilibrium value $\mathbf{B}_{p0}$. Then, with the use of $\mathbf{A}_{p1} = \mathbf{A}_{p0} + \delta\mathbf{A}_p$, equation (17) becomes

$$\int_{C_{p1}} \mathbf{A}_{p1} \cdot d\mathbf{C} - \int_{C_{p0}} \mathbf{A}_{p0} \cdot d\mathbf{C} = \int_{C_{p0}} (\delta\mathbf{A}_p - \delta\mathbf{a} \times \mathbf{B}_{p0}) \cdot d\mathbf{C}. \quad (19)$$

But from the invariance of the toroidal flux in the plasma it follows that the left-hand side of equation (19) vanishes. Consequently,

$$\int_{C_{p0}} (\delta\mathbf{A}_p - \delta\mathbf{a} \times \mathbf{B}_{p0}) \cdot d\mathbf{C} = 0. \quad (20)$$

Another example of a contour inside of which the magnetic flux is conserved (it being zero) is a closed curve on the plasma surface that encircles the torus neither the long way nor the short way. For this contour, again equation (20) holds. The necessary and sufficient condition for equation (20) to hold for arbitrary contours in the plasma surface is

$$\delta\mathbf{A}_p = \delta\mathbf{a} \times \mathbf{B}_{p0}. \quad (21)$$

The vector potential $\mathbf{A}$ on the plasma–vacuum boundary is continuous. By choosing contours that encircle the vacuum region in the perturbed state and in the equilibrium state respectively, we can show that a relation analogous to equation (21) holds at the vacuum side of the plasma boundary:

$$\delta\mathbf{A}_v = \delta\mathbf{a} \times \mathbf{B}_{v0}. \quad (22)$$

The relations (21) and (22) also follow from the condition (locally valid both on the equilibrium and perturbed plasma surface):

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = 0.$$

Taking into account that in equilibrium both $\mathbf{E}$ and $\mathbf{v}$ vanish, we obtain for the perturbations the relation

$$\delta\mathbf{E} + \delta\mathbf{v} \times \mathbf{B} = 0,$$

with $\delta\mathbf{E} = -\delta_t\delta\mathbf{A}$ and $\delta\mathbf{v} = \delta_t\delta\mathbf{a}$, this leads directly to the equations (21) and (22).

On the metal wall, the variation $\delta\mathbf{A}(\mathbf{r}_m)$ vanishes. Using this and equations (9), (20) and (21) in the surface integrals of equation (16), we find that they reduce to zero. This results in the Euler equations

$$\nabla \times \mathbf{B}_p = \lambda\mathbf{B}_p, \quad (23)$$

and

$$\nabla \times \mathbf{B}_v = 0. \quad (24)$$

Thus, next to the trivial vacuum equation (24), we find again the Euler equation for the field in the plasma, already derived by WOLTJER (1958).

3. THE ONE-DIMENSIONAL (CYLINDRICAL) SOLUTION

Below we give a listing of all the relevant physical quantities, as they follow from the Euler equations, boundary conditions and values of the invariants. They are valid for a torus in the limit of the aspect ratio R/a going to infinity. The resulting cylindrical equilibrium is assumed to have translational and rotational symmetry. The subscripts p for poloidal and t for toroidal have been replaced by ϑ and z respectively. It has been shown (REIMAN, 1981) that one may always choose $\psi_h = 0$, so that in the cylindrical limit $A_z(r_m) = 0$.

We have introduced the dimensionless quantities $x = \lambda a$ and $\rho = a/r_m$, where a is the plasma radius and r_m is the radius of the metal wall.

$$B_{p\vartheta} = \frac{\psi_{pz}}{\pi r_m^2 \rho^2} \frac{x}{2J_1(x)} J_1\left(\frac{x}{\rho r_m} r\right), \quad (25a)$$

$$B_{pz} = \frac{\psi_{pz}}{\pi r_m^2 \rho^2} \frac{x}{2J_1(x)} J_0\left(\frac{x}{\rho r_m} r\right), \quad (25b)$$

$$A_{p\vartheta} = \frac{\psi_{pz}}{\pi r_m \rho} \frac{1}{2J_1(x)} J_1\left(\frac{x}{\rho r_m} r\right), \quad (25c)$$

$$A_{pz} = \frac{\psi_{pz}}{\pi r_m \rho} \frac{1}{2J_1(x)} \left\{ J_0\left(\frac{x}{\rho r_m} r\right) - J_0(x) \right\}. \quad (25d)$$

$$B_{v\vartheta} = -\frac{\psi_{v\vartheta}}{2\pi R \log \rho} \frac{1}{r}, \quad (26a)$$

$$B_{vz} = \frac{\psi_{vz}}{\pi r_m^2} \frac{1}{1 - \rho^2}, \quad (26b)$$

$$A_{v\vartheta} = \frac{\psi_{vz}}{2\pi r_m^2} \frac{r}{1 - \rho^2} + \frac{1}{\pi} \left(\frac{1}{2} \psi_{pz} - \frac{\rho^2 \psi_{vz}}{1 - \rho^2} \right) \frac{1}{r}, \quad (26c)$$

$$A_{vz} = \frac{\psi_{v\vartheta}}{2\pi R} \frac{\log r/a}{\log \rho}. \quad (26d)$$

In addition to this solution, there are two equations from which x (or λ) and ρ (the plasma position) are to be determined. They follow because the number of boundary conditions and constraints exceeds by two the number of integration constants in the solutions of the Euler equations.

$$Q_1 = \frac{x}{\rho} \left\{ 1 + \frac{J_0^2(x)}{J_1^2(x)} - \frac{2J_0(x)}{xJ_1(x)} \right\}, \quad (27)$$

$$Q_2^2 = \frac{\log^2 \rho}{\rho^2} \left\{ \frac{x^2}{4} \left(1 + \frac{J_0^2(x)}{J_1^2(x)} \right) \frac{\rho^4 Q_2^2}{(1 - \rho^2)^2} \right\}, \quad (28)$$

where the following abbreviations have been used:

$$Q_1 = \frac{r_m}{R} \frac{\mu_0 K - \psi_{pz} \psi_{v\theta}}{\psi_{pz}^2}, \quad (29)$$

$$Q_2 = \frac{\psi_{vz}}{\psi_{pz}}, \quad (30)$$

$$Q_3 = \frac{r_m \psi_{v\theta}}{2R \psi_{pz}}. \quad (31)$$

In the limit of the plasma touching the metal wall, Taylor's solution is recovered. In this limit, $Q_2 = Q_3 = 0$ and equation (28) has the solution $\rho = 1$, or $a = r_m$. Then $x = \lambda a$ follows from equation (27).

4. DISCUSSION OF THE CYLINDRICAL SOLUTION

The pair of equations (27) and (28) will in general have a set of roots (x_i, ρ_i) . In the Taylor limit $\rho \rightarrow 1$, REIMAN (1981) and TAYLOR (1974) came to the conclusion that the smallest root x_i minimizes the magnetic energy of the plasma. In the presence of a vacuum layer the situation is more complex, but again the root x_i has to be sought that minimizes the total magnetic energy. For the magnetic energy of a plasma reaching to the metal wall, REIMAN [1981, equations (5) and (13)] derived the expression (in units with $\mu_0 = 1$)

$$2W_{Bp} = \lambda K_0 - \lambda \psi_h \psi_{pt} + I_h \psi_{pt}, \quad (32)$$

where I_h is the (poloidal) current through the hole of the torus, being the sum of the current through the external field coil windings and the induced poloidal current in the metal wall. For the present case of a plasma with a vacuum layer, equation (32) must be extended to become

$$2W_{Bp} = \lambda K_0 - \lambda(\psi_h + \psi_{vp})\psi_{pt} + (I_h + i_{pp})\psi_{pt}, \quad (33)$$

where i_{pp} is the poloidal surface current in the plasma boundary. The magnetic field in the vacuum can be written as the gradient of a multivalued function Φ :

$$\mathbf{B}_v = \nabla \Phi. \quad (34)$$

Then the magnetic energy density $B_v^2 = \nabla \cdot \Phi \mathbf{B}_v$ and the magnetic energy in the vacuum layer becomes:

$$2W_{Bv} = \int_{V_v} B_v^2 dV = \int_{S_v} \Phi \mathbf{B}_v \cdot d\mathbf{S}. \quad (35)$$

The closed surface S_v consists of the surface of the metal wall, the surface of the plasma-vacuum interface, the poloidal cut in the vacuum S_{vpc} and the toroidal cut in the vacuum S_{vtc} . Only the cuts will contribute to the integral in equation (35):

$$2W_{Bv} = \int_{S_{vpc}} [\Phi]_r \mathbf{B}_v \cdot d\mathbf{S} + \int_{S_{vtc}} [\Phi]_\theta \mathbf{B}_v \cdot d\mathbf{S}. \quad (36)$$

The toroidal jump in Φ , $[\Phi]_t$, is calculated by integration of $\nabla\Phi$ along a closed toroidal contour in the vacuum:

$$[\Phi]_t = \int_{C_{vt}} \nabla\Phi \cdot d\mathbf{C} = \int_{C_{vt}} \mathbf{B}_v \cdot d\mathbf{C} = \int_{S_{vt}} \mathbf{j} \cdot d\mathbf{S} = I_h. \quad (37)$$

The poloidal jump $[\Phi]_p$ is calculated likewise by integration of $\nabla\Phi$ along any closed poloidal contour around the plasma column.

$$[\Phi]_p = \int_{C_{vp}} \nabla\Phi \cdot d\mathbf{C} = \int_{C_{vp}} \mathbf{B}_v \cdot d\mathbf{C} = \int_{S_{vp}} \mathbf{j} \cdot d\mathbf{S} = I_{pt} + i_{pt} = I_t, \quad (38)$$

where i_{pt} is the poloidal surface current on the plasma. Substituting the jumps in equation (36) we find

$$2W_{Bv} = I_h\psi_{vt} + I_t\psi_{vp}. \quad (39)$$

The total magnetic energy W_B is found by combination of equations (33) and (39).

$$2W_B = \lambda K_0 - \lambda(\psi_h + \psi_{vp})\psi_{pt} + (I_h + i_{pp})\psi_{pt} + I_h\psi_{vt} + I_t\psi_{vp}. \quad (40)$$

In view of $I_{pt} = \lambda\psi_{pt}$ and by introducing $\psi_t = \psi_{pt} + \psi_{vt}$, we can write equation (40) as

$$2W_B = \lambda K_0 - \lambda\psi_h\psi_{pt} + I_h\psi_t + i_{pp}\psi_{pt} + i_{pt}\psi_{vp}. \quad (41)$$

In the absence of surface currents it can be seen that the lowest λ corresponds to the lowest value of the magnetic energy. In the complicated case with surface currents this is no longer true.

5. DISCUSSION

From solutions (25) and (26) the familiar F - ϑ -diagram can be constructed. In the presence of a vacuum layer the relevant definitions are:

$$F = \frac{B_z(r_m)}{\bar{B}_z} \text{ and } \vartheta = \frac{B_\vartheta(r_m)}{\bar{B}_z}, \text{ where } \bar{B}_z = \frac{\psi_t}{\pi r_m^2} \text{ is now the average}$$

over the total cross-section. When there are no surface currents, we find

$$F = \frac{\vartheta r_m}{a} J_0 \left[\frac{2\vartheta a/r_m}{1 - F\{1 - (a/r_m)^2\}} \right] / J_1 \left[\frac{2\vartheta a/r_m}{1 - F\{1 - (a/r_m)^2\}} \right]. \quad (42)$$

For $a = r_m$, equation (42) reduces to Taylor's result: $F(\vartheta, r_m) = \vartheta J_0(2/J_1(2\vartheta))$, with $F = 0$ at $\vartheta \approx 1.202$.

Furthermore, we see from equation (42) that field reversal takes place for $\vartheta a/r_m > 1.2$. Consequently, a vacuum layer will effect a rightward shift of the $F - \vartheta$ -curve. This is shown in Fig. 3, where $F - \vartheta$ -curves have been drawn for several values of a/r_m .

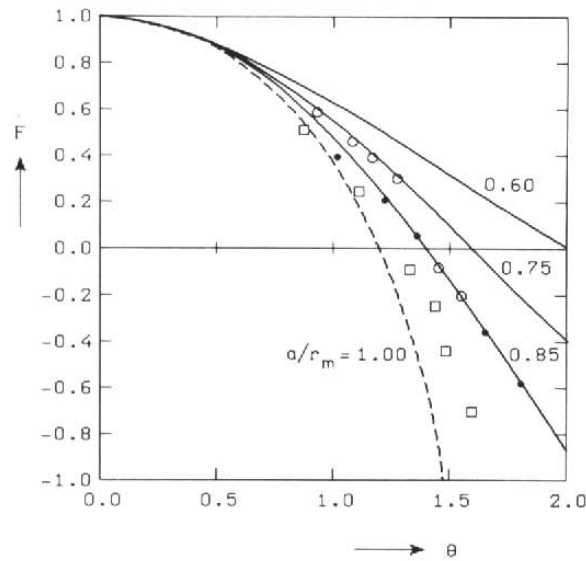


FIG. 3.— F - θ -diagram for various positions of the plasma-vacuum boundary.

The data symbols in this plot referring to different filling pressures were taken from EDENSTRASSER and SCHUURMAN (1984). The conclusion is qualitatively that besides a finite β the presence of a vacuum layer can be used to explain the experimental points to the right of the Taylor curve (dashed in Fig. 3).

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